§3.4 Limits at Infinity.

Key points: ① horizontal/vertical asymptotes; $\lim_{x\to\pm\infty} f(x)=L$ and $\lim_{x\to a^\pm} f(x)=\pm\infty$.

② $\frac{1}{0^\pm}=\pm\infty$, $\frac{1}{\pm\infty}=0$, $\infty^{\text{positive power}}=\infty$, $\infty^{\text{negative power}}=0$.

③ Highest term (leading term) rule for $\lim_{x\to\pm\infty}$

• Def: $\lim_{x\to\infty} f(x)=L$ $(x\to-\infty)$ means as x approaches infinity $(+\infty$ or $-\infty)$ (as x gets arbitrarily large (positive or negative)) $f(x)$ approaches L.

If L is finite, $y=L$ is called a horizontal asymptote of $y=f(x)$.

Recall: If $\lim_{x\to a^\pm} f(x)=\pm\infty$, $x=a$ is called a vertical asymptote of $y=f(x)$. (Sec 1.5, lec note week 1. page 5).

• $x\to\infty$ can be treated as "finite numbers" following the rules below:

① $\lim_{x\to\pm\infty}\frac{1}{x}=0 \iff$ "$\frac{1}{\pm\infty}=0$". In §1.5, we have $\lim_{x\to0^\pm}\frac{1}{x}=\pm\infty \iff$ "$\frac{1}{0^\pm}=\pm\infty$".

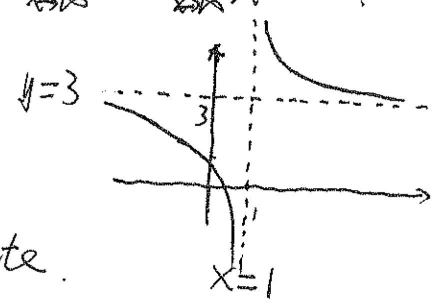
② $x^{\text{positive power}}$ approaches ∞ as x approaches ∞: $\lim_{x\to\infty}\sqrt{x}=\infty$, $\lim_{x\to\infty}x=\infty$, $\lim_{x\to\infty}x^{\frac{3}{2}}=\infty$, $\lim_{x\to\infty}x^2=\infty$

$x^{\text{negative power}}=\frac{1}{x^{\text{positive power}}}\xrightarrow{x\to\infty}\infty$: $\lim_{x\to\infty}x^{-\frac12}=\lim_{x\to\infty}\frac{1}{\sqrt{x}}=0$, $\lim_{x\to\infty}x^{-5}=\lim_{x\to\infty}\frac{1}{x^5}=0$. ......

eg.1. $y=3+\frac{2}{x-1}$. $\lim_{x\to\pm\infty}3+\frac{2}{x-1}=3+\frac{2}{\pm\infty}=3$

(sec 1.5 $\Rightarrow$) $\lim_{x\to1^+}3+\frac{2}{x-1}=\infty$, $\lim_{x\to1^-}3+\frac{2}{x-1}=-\infty$.

$y=3$ is a horizontal asymptote and $x=1$ is a vertical asymptote.



Remark: $\frac{\infty}{\infty}$ or $\infty-\infty$ is indeterminate, we have to do some algebra changes first.

• Highest term (leading term) rule: In order to evaluate the limits for a ratio of power functions, we only need to keep the highest order terms in the numerator and the denominator and DROP ALL THE LOWER ORDER TERMS.

eg.2. $\lim_{x\to\infty}\frac{2-3x^2}{3+2x+5x^2}=\lim_{x\to\infty}\frac{-3x^2}{5x^2}=\lim_{x\to\infty}\frac{-3}{5}=-\frac{3}{5}$. $y=-\frac{3}{5}$ horizontal asymptote.

Remark: $-3x^2$ is the highest term in the numerator; $5x^2$ is the highest term in the denominator.

eg.3. (More examples about highest term rule).

- $\lim_{x\to-\infty} \frac{-7x+\sqrt{x}}{x^3+2x} = \lim_{x\to\infty} \frac{-7x}{x^3} = \lim_{x\to\infty} \frac{-7}{x^2} = \left(\frac{-7}{\infty}\right) = 0$

- $\lim_{x\to\infty} \frac{2+3\cdot x^{\frac{3}{2}}}{1-\sqrt{x}} = \lim_{x\to\infty} \frac{3\cdot x^{\frac{3}{2}}}{-x^{\frac{1}{2}}} = \lim_{x\to\infty} -3\cdot x^1 = -\infty$. Remark: $\frac{x^a}{x^b} = x^{a-b} = \frac{1}{x^{b-a}}$

- $\lim_{x\to\infty} \frac{5x}{3-2x} = \lim_{x\to\infty} \frac{5x}{-2x} = -\frac{5}{2}$. $y=-\frac{5}{2}$ is the horizontal asymptote.

Remark: Highest order rule is only applied to $x\to\infty$.

$\lim_{x\to(\frac{3}{2})^+} \frac{5x}{3-2x} \underset{\text{Direct plug in}}{=} \frac{5\cdot\frac{3}{2}}{3-2\cdot\frac{3}{2}} = \frac{\text{finite number}}{0^-} = -\infty$. $x=\frac{3}{2}$ is vertical asymptote

(negative sign comes from $x\to(\frac{3}{2})^+$
$x>\frac{3}{2} \Rightarrow 3-2x<0$.)

Remark: Highest order rule has following product form.

$\lim_{x\to\infty} \frac{(2-6x)\cdot(x^2+1)}{(3x+1)\cdot(2x^2-x)} = \lim_{x\to\infty} \frac{(-6x)\cdot x^2}{3x\cdot 2x^2}$. Pick the highest term in each bracket.

$= \lim_{x\to\infty} \frac{-6x^3}{6x^3} = -1$.

Remark: The formal argument for highest term rule: Pull out the highest order terms.

eg.4 (Re-prove eg.2).

$$\lim_{x\to\infty} \frac{2-3x^2}{3+2x+5x^2} = \lim_{x\to\infty} \frac{\cancel{x^2}\cdot(\frac{2}{x^2}-3)}{\cancel{x^2}\cdot(\frac{3}{x^2}+\frac{2x}{x^2}+5)} = \frac{0-3}{0+0+5} = -\frac{3}{5}.$$

Hints for WW.

※5, ※6: Vertical asymptote. See more examples in §1.5, 1.6. LecNotes Week1: Page 5-6.

※7: Conjugation for root: $\lim_{x\to\infty} \sqrt{x^2+3x}-x = \lim_{x\to\infty} \frac{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)}{\sqrt{x^2+3x}+x} = \lim_{x\to\infty} \frac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}+x}$

$= \lim_{x\to\infty} \frac{3x}{\sqrt{x^2}+x}$ ← highest order rule for x^2+3x.

$= \lim_{x\to\infty} \frac{3x}{x+x} = \frac{3}{2}$.

$\lim_{x\to\infty} \sqrt{4x+1}-4x = \lim_{x\to\infty} \frac{(\sqrt{4x+1}-4x)(\sqrt{4x+1}+4x)}{\sqrt{4x+1}+4x}$.

$= \lim_{x\to\infty} \frac{4x+1-16x^2}{\sqrt{4x+1}+4x} = \lim_{x\to\infty} \frac{-16x^2}{4x}$.

$= \lim_{x\to\infty} -4x = -\infty$.

※8. (Squeeze theorem §1.6)

$\frac{-1+x}{x} \le \frac{\sin x+x}{x} \le \frac{1+x}{x}$ since $-1\le\sin x\le 1$. $\lim_{x\to\infty}\frac{1+x}{x}=1$, $\lim_{x\to\infty}\frac{-1+x}{x}=1 \Rightarrow \lim_{x\to\infty}\frac{\sin x+x}{x}=1$.

§3.5. Curve Sketching

key points: ① Polynomial long division

② Slant asymptote for rational functions.

③ (Curve sketching) combination of 3.3, 3.4, 3.5.

eg.0. • Divide 17 by 5, we have $17 = 3\cdot 5 + 2$

$$\begin{array}{r} 3 \leftarrow \text{quotient} \\ 5\,\overline{)\,17} \\ \underline{15} \\ 2 \leftarrow \text{remainder.} \end{array}$$

• Divide x^2+2x-4 by $x-1$, $x^2+2x-4 = q(x)\cdot(x-1)+r(x)$

Find the quotient $q(x)$ and remainder $r(x)$ by Polynomial long division.

$$\begin{array}{r} x+3 \leftarrow q(x) \\ x-1\,\overline{)\,x^2+2x-4} \\ \underline{x^2-x\quad} \\ 3x-4 \\ \underline{3x-3} \\ -1 \leftarrow r(x) \end{array}$$

i.e. $x^2+2x-4 = (x+3)\cdot(x-1) - 1$.

• Consider the ratio $\frac{17}{5} = \frac{3\cdot 5+2}{5} = 3 + \frac{2}{5}$

• Consider the ratio of polynomials: $\frac{x^2+2x-4}{x-1} = \frac{(x+3)\cdot(x-1)-1}{x-1} = x+3 - \frac{1}{x-1}$.
(Rational functions)

• Slant asymptote: If $f(x)$ approaches a line $y = m\cdot x + b$ as x approaches infinity, then $y = mx+b$ is the SLANT ASYMPTOTE of $f(x)$.

eg1: $f(x) = \frac{x^2+2x-4}{x-1} = \boxed{x+3} - \frac{1}{x-1}$. $f(x)$ approaches $y=x+3$ as $x\to\infty$

since $\| f(x) - (x+3) \| = -\frac{1}{x-1} \to 0$ as $x\to\infty$

i.e. $y = x+3$ is the slant asymptote of $f(x)$.

• Conclusion: If a rational function can be written as $f(x) = m\cdot x + b + \frac{r(x)}{d(x)}$ via polynomial long division, then $y = mx+b$ is the slant asymptote of $y = f(x)$.

eg.2. Let $f(x) = \frac{4x^2}{2x-5}$. Find all the asymptotes (vertical/horizontal/slant) of $f(x)$.

- $\boxed{\text{Vertical: } x = \frac{5}{2}}$ since $\lim_{x\to(\frac{5}{2})^+} \frac{4x^2}{2x-5} = \infty$. (or $\lim_{x\to(\frac{5}{2})^-} \frac{4x^2}{2x-5} = -\infty$).

- $\boxed{\text{Horizontal: None.}}$ $\lim_{x\to\pm\infty} \frac{4x^2}{2x-5} \overset{\text{highest term}}{=\!=} \lim_{x\to\pm\infty} \frac{4x^2}{2x} = \lim_{x\to\pm\infty} 2x = \pm\infty$ (Not finite)

- $\boxed{\text{Slant: } y = 2x+5.}$ Poly-Long Division:

$$
\begin{array}{r|l}
 & \ \ 2x + 5 \\ \hline
2x-5 & 4x^2 + 0 + 0 \\
 & \underline{4x^2 - 10x} \\
 & \qquad 10x + 0 \\
 & \qquad \underline{10x - 25} \\
 & \qquad\qquad 25
\end{array}
$$

$4x^2 = \underbrace{(2x+5)}_{\text{quotient}}\cdot(2x-5) + \underbrace{25}_{\text{remainder}}$

$\frac{4x^2}{2x-5} = \underbrace{2x+5}_{\text{slant asymp.}} + \frac{25}{2x-5}$

since $\frac{4x^2}{2x-5} = 2x+5 + \frac{25}{2x-5}$.

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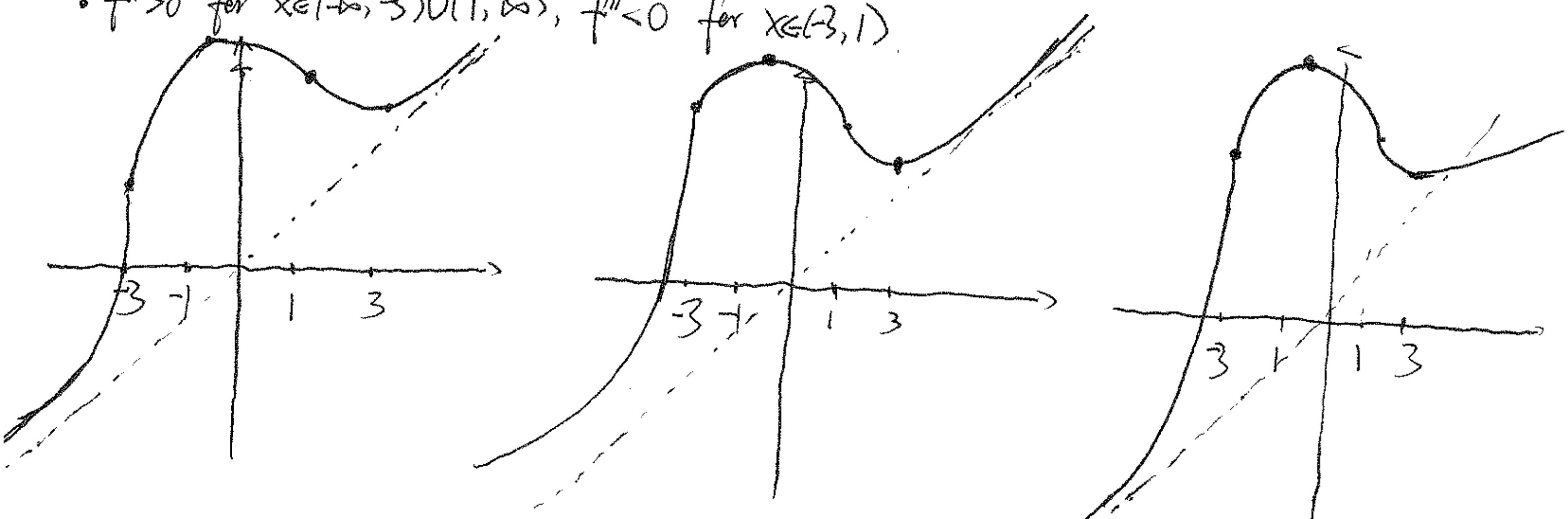
eg.3. (Curve Sketching: Combination of 3.3-3.5, f16)

Sketch the graph of $y=f(x)$ such that (The answer is not unique)

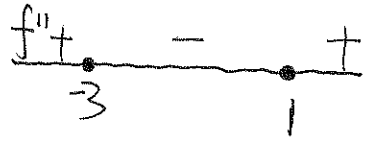
- f is continuous and has SLANT asymptote $y=x$
- $f'>0$ for $x\in(-\infty,-1)\cup(3,+\infty)$, $f'<0$ for $x\in(-1,3)$
- $f''>0$ for $x\in(-\infty,-3)\cup(1,\infty)$, $f''<0$ for $x\in(-3,1)$

All the following three are qualified answers.



- Local maximum of f occurs at $x=-1$.
- Local minimum of f occurs at $x=3$

- Inflection points of f are $x=-3$, $x=1$



eg 4. Suppose $f(x)=\frac{x}{x^2+1}$, $f'(x)=\frac{1-x^2}{(x^2+1)^2}$, $f''(x)=\frac{2(x^3-3x)}{(x^2+1)^3}$

(f1b).

(a). f is an odd function whose graph is symmetric with respect to the origin

Reason: $f(-x)=\frac{-x}{(-x)^2+1}=-\left[\frac{x}{x^2+1}\right]=-f(x)$.

Remark: f is even if $f(-x)=f(x)$

f is odd if $f(-x)=-f(x)$.

(b). Interval of increasing/decreasing and local extremums.

$f'(x)=\frac{1-x^2}{(x^2+1)^2}=\frac{(1-x)(1+x)}{(x^2+1)^2}=0 \Rightarrow x=-1, 1$ (two critical points)

(defined for all x)

$-1, 1$ divide $(-\infty,+\infty)$ into

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,+\infty)$
	$x<-1$		$-1<x<1$		$x>1$
f'	$---$		$+++$		$---$
	$f'(-2)<0$		$f'(0)>0$		$f'(2)<0$.

Increasing: $\cancel{(-1,1)}$ $[-1,1]$ where $f'>0$

Decreasing: $(-\infty,-1]\cup[1,+\infty)$ where $f'<0$

local maximum occurs at $x=1$, local minimum occurs at $x=-1$.

(c). Concavity: $f''(x)=\frac{2x\cdot(x^2-3)}{(x^2+1)^3}=\frac{2x\cdot(x+\sqrt{3})\cdot(x-\sqrt{3})}{(x^2+1)^3}=0$

$\Rightarrow x=0,\ x=-\sqrt{3},\ x=\sqrt{3}$

x		$-\sqrt{3}$		0		$\sqrt{3}$	
f''	$---$		$+++$		$---$		$+++$
	$f''(-2)<0$		$f''(-1)>0$		$f''(1)<0$		$f''(2)>0$

Concave up: $(-\sqrt{3},0)\cup(\sqrt{3},+\infty)$
($f''>0$)

Concave down: $(-\infty,-\sqrt{3})\cup\cancel{(\sqrt{3},+\infty)}$ $(0,\sqrt{3})$.
($f''<0$)

Inflection points: $x=-\sqrt{3}, 0, \sqrt{3}$.

Hints for WW 6. $f(x)=\frac{x^3}{x^2-9}$, $f'(x)=\frac{x^4-27x^2}{(x^2-9)^2}$, $f''(x)=\frac{18x\cdot(x^2+27)}{(x^2-9)^3}$

(You can seek help from WolframAlpha for the expression of f'').

x		$-\sqrt{27}$		-3		0		3		$\sqrt{27}$	
f'	$+$		$---$				$---$				$+++$

Increasing: $(-\infty,-\sqrt{27}]\cup[\sqrt{27},+\infty)$

Decreasing: $[-\sqrt{27},-3)\cup(-3,3)\cup(3,\sqrt{27}]$

Rmk: -3, 3 are not in the domain.

x		-3		0		3	
f''	$---$		$+++$		$---$		$+++$
	$f''(-4)<0$		$f''(-1)>0$		$f''(1)<0$		$f''(4)>0$

Concave up: $(-3,0)\cup(3,+\infty)$

Concave down: $(-\infty,-3)\cup(0,3)$

Inflection: $x=0$. ($x=\pm3$ not in the domain)

☆☆☆ eg 5. Analyze $f(x) = 2x - 3\cdot x^{\frac{2}{3}}$ (related to ww 5).

① Domain of f : $(-\infty, \infty)$. $\Rightarrow$ f has no vertical asymptote

② f has no horizontal asymptote since $\lim\limits_{x\to\pm\infty} 2x - 3\cdot x^{\frac{2}{3}} = \lim\limits_{x\to\pm\infty} x\left[2 - \frac{3}{x^{\frac{1}{3}}}\right]$

③ $f'(x) = (2x - 3\cdot x^{\frac{2}{3}})' = \boxed{2 - 2\cdot x^{-\frac{1}{3}}}$ $\qquad = \pm\infty\cdot(2-0) = \pm\infty$

④ Critical points of f : (Hint: critical points $\Leftrightarrow$ f' D.N.E or $f'=0$)

$f'(x) = 2 - \frac{2}{x^{\frac{1}{3}}}$ D.N.E $\Leftrightarrow$ Denominator is zero $\Leftrightarrow$ $x^{\frac{1}{3}} = 0 \Rightarrow x = 0$.

$f'(x) = 2 - \frac{2}{x^{\frac{1}{3}}} = 0 \Rightarrow 2 = \frac{2}{x^{\frac{1}{3}}} \Rightarrow x^{\frac{1}{3}} = 1 \Rightarrow x = 1$.

$\boxed{\text{Critical points are } x=0 \text{ and } x=1}$

⑤ Increasing/Decreasing Intervals : (determined by the signs of f').

critical points $0, 1$ divide $(-\infty, \infty)$ into $(-\infty, 0)$ 0 $(0,1)$ 1 $(1, +\infty)$

	$x<0$	$0<x<1$	$x>1$
f'	$+++$	$---$	$+++$

$x<0$, $f'(x) > 0$ $\quad (f'(-1) = 4 > 0)$

$0<x<1$, $\quad 0 < x^{\frac{1}{3}} < 1 \Rightarrow \quad 2 - \frac{2}{x^{\frac{1}{3}}} < 0$, $\quad f' < 0$

$x>1$, $\quad x^{\frac{1}{3}} > 1 \Rightarrow \quad 2 - \frac{2}{x^{\frac{1}{3}}} > 0$, $\quad f' > 0$.

Increasing Interval(s) : $(-\infty, 0] \cup [1, +\infty)$. Decreasing Interval(s) : $[0, 1]$

⑥ local maximum : attained at $x=0$ (increasing $\to$ decreasing)

local minimum : attained at $x=1$ (decreasing $\to$ increasing)

⑦ Concavity and inflection points : $f''(x) = (2 - 2\cdot x^{-\frac{1}{3}})' = 0 - 2\cdot(-\frac{1}{3})x^{-\frac{2}{3}} = \frac{2}{3}\cdot x^{-\frac{2}{3}}$

Notice that $f'' = \frac{2}{3}\cdot\frac{1}{(x^{\frac{1}{3}})^2}$ is forever positive except 0 since $\square^2 > 0$

f is concave up on $(-\infty, 0) \cup (0, +\infty)$

and concave down nowhere

and no inflection points.